

# Indicated coloring of matroids

Michał Lason

**ABSTRACT.** A coloring of a matroid is *proper* if elements of the same color form an independent set. For a loopless matroid  $M$ , its chromatic number  $\chi(M)$  is the minimum number of colors that suffices to color properly the ground set  $E$  of  $M$ . In this note we study a game-theoretic variant of this parameter proposed by Grytczuk. Suppose that in each round of the game Alice indicates an uncolored yet element  $e$  of  $E$ , then Bob colors it using a color from a fixed set of colors  $C$ . The rule Bob has to obey is that it is a proper coloring. The game ends if the whole matroid has been colored or if Bob can not color  $e$  using any color of  $C$ . Alice wins in the first case, while Bob in the second. The minimum size of the set of colors  $C$  for which Alice has a winning strategy is called the *indicated chromatic number* of  $M$ , denoted by  $\chi_i(M)$ . We prove that  $\chi_i(M) = \chi(M)$ .

## 1. Introduction

Let  $M$  be a loopless matroid on a ground set  $E$  (the reader is referred to [10] for background of matroid theory). In analogy to the graph case we say that a coloring of the set  $E$  is *proper* if elements of the same color form an independent set of  $M$ . The *chromatic number* of  $M$ , denoted by  $\chi(M)$ , is the minimum number of colors that suffice to color properly the set  $E$ . In case of a graphic matroid  $M = M(G)$ , the number  $\chi(M)$  is a well studied parameter known as the *arboricity* of the underlying graph  $G$ .

The study of game-theoretic variants of chromatic number was initiated for graphs independently by Brams, cf. [5], and Bodlaender [3]. They defined game chromatic number of a graph, which was intensively studied (see [1, 2, 4, 7, 13]). A natural question concerning all game-theoretic variants of chromatic number is whether it is bounded from above by a function of chromatic number, and if yes then what is the best possible bound. The game chromatic number of a graph is not bounded, as it can be arbitrary large for bipartite graphs. A matroidal version—the game chromatic number  $\chi_g(M)$  of a matroid was studied in [8], where the author shows that  $\chi_g(M) \leq 2\chi(M)$  for every matroid  $M$ . This gives a nearly tight bound, since for every  $k \geq 3$  there are matroids with  $\chi(M) = k$  and  $\chi_g(M) \geq 2k - 1$ .

Another game-theoretic variant of list chromatic number was introduced by Schauz [11] (see also [14]), it is called on-line list chromatic number. For graphs it

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is bounded by an exponential function of chromatic number, and there are known examples for which both parameters differ. For matroids they are always equal [9], and also equal to the chromatic number [12].

The newest variant of the graph coloring game was proposed by Grytczuk. Let  $G$  be a graph, and let  $C$  be a fixed set of colors. In each round of the game Alice indicates an uncolored yet vertex, then Bob colors it using a color from  $C$ . The only rule Bob has to obey is that it is a proper coloring. The goal of Alice is to achieve a proper coloring of the whole graph, while Bob is trying to prevent it (arrive at a partial coloring that can not be extended). The minimum size of the set of colors  $C$  for which Alice has a winning strategy is called the *indicated chromatic number* of a graph  $G$ , denoted by  $\chi_i(G)$ . Clearly  $\chi_i(G) \geq \chi(G)$ . Grzesik [6] proved that if  $\chi(G) = 2$ , then  $\chi_i(G) = 2$  and gave an example of a graph  $G$  with  $\chi(G) = 3$  and  $\chi_i(G) = 4$ . He also shows an upper bound  $\chi_i(G) \leq 4\chi(G)$  for random graphs, and conjectures that indicated chromatic number of a graph is bounded by a function of chromatic number.

In this paper we study a matroidal version of the indicated chromatic number. Our main result reads as follows.

**THEOREM 1.** *Every loopless matroid  $M$  satisfies  $\chi_i(M) = \chi(M)$ .*

The proof is by induction used to a suitable generalization of the game. We end the paper with a fancy modification of indicated chromatic number also made by Grytczuk.

## 2. Indicated chromatic number

The main tool we use is the matroid union theorem (for a proof see [10]).

**THEOREM 2.** (Matroid Union Theorem) *Let  $M_1, \dots, M_k$  be matroids on the same ground set  $E$ , with rank functions  $r_1, \dots, r_k$  respectively. The following conditions are equivalent:*

- (1) *there exist sets  $V_i$  with  $V_1 \cup \dots \cup V_k = E$ , such that for each  $i$  the set  $V_i$  is independent in  $M_i$ ,*
- (2) *for each  $A \subset E$  holds  $r_1(A) + \dots + r_k(A) \geq |A|$ .*

For seek of completeness we repeat the definition of the game in a matroid setting. Let  $M$  be a matroid on a ground set  $E$ . In each round of the game Alice indicates an uncolored yet element of  $E$ , then Bob colors it using a color from a fixed set of colors  $C$ . The rule Bob has to obey is that it is a proper coloring. The game ends if the whole matroid has been colored or if Bob can not color  $e$  using any color of  $C$ . Alice wins in the first case, while Bob in the second. The *indicated chromatic number* of a matroid  $M$ , denoted by  $\chi_i(M)$ , is the minimum size of the set of colors  $C$  for which Alice has a winning strategy. Clearly  $\chi_i(M) \geq \chi(M)$ .

Theorem 1 is a corollary of a slightly more general Theorem 3 which concerns a game explained below.

Let  $M_1, \dots, M_k$  be a collection of matroids on the same ground set  $E$ . We consider a modification of the game in which the set of colors is  $\{1, \dots, k\}$ . The other difference concerns the rule Bob has to obey. Namely elements of color  $i$  must form an independent set in a matroid  $M_i$ . As usual, Alice wins if the whole set  $E$  has been colored, while Bob wins if a partial coloring can not be properly extended.

**THEOREM 3.** *Let  $M_1, \dots, M_k$  be matroids on the same ground set  $E$ . Suppose there are sets  $V_1, \dots, V_k$ , such that  $V_i$  is independent in  $M_i$ , and  $V_1 \cup \dots \cup V_k = E$ . Then Alice has a winning strategy in the generalized indicated coloring game.*

**PROOF.** The proof goes by induction on the number of elements of  $E$ . For a set  $E$  consisting of only one element assertion clearly holds. Thus we assume that  $|E| > 1$  and that the assertion is true for all smaller sets.

Let us denote by  $M|_A$  the restriction of a matroid  $M$  to a set  $A$ , and by  $M/A$  the contraction of a set  $A$  in a matroid  $M$ . Denote also the rank functions of matroids  $M_1, \dots, M_k$  respectively by  $r_1, \dots, r_k$ . From Theorem 2 we know that for each  $\emptyset \neq A \subset E$  holds an inequality

$$r_1(A) + \dots + r_k(A) \geq |A|.$$

There are two cases:

Case 1: there is a proper subset  $\emptyset \neq A \subsetneq E$  with equality

$$r_1(A) + \dots + r_k(A) = |A|.$$

Then by subtracting equality for  $A$  from the inequality for  $A \cup B$  we get that for every subset  $B \subset E \setminus A$  holds

$$r_1(A \cup B) - r_1(A) + \dots + r_k(A \cup B) - r_k(A) \geq |A \cup B| - |A| = |B|.$$

The left side of this inequality is the sum of ranks of the set  $B$  in matroids  $M_1/A, \dots, M_k/A$ . Thus by Theorem 2 the collection of matroids  $M_1/A, \dots, M_k/A$  on  $E \setminus A$  satisfies assumptions of the theorem. The collection  $M_1|_A, \dots, M_k|_A$  of matroids on  $A$  clearly also does. By inductive assumption Alice has a winning strategy in the game with matroids  $M_1|_A, \dots, M_k|_A$  on the set  $A$ , so she plays with this strategy. Let us denote elements colored by Bob with  $i$  after the game is finished by  $U_i$ . Now the play moves to the set  $E \setminus A$ , so now the original game is on matroids  $M_i/U_i$ . But since the collection of matroids  $M_i/A$  satisfies assumptions of the theorem, collection of matroids  $M_i/U_i$  also does and Alice has a winning strategy by inductive assumption. As a result she wins the whole game on  $E$ .

Case 2: for all  $\emptyset \neq A \subsetneq E$  holds

$$r_1(A) + \dots + r_k(A) > |A|.$$

Then in the first round of the game Alice indicates an arbitrary element  $e \in E$ . Obviously  $e \in V_l$  for some  $l$ , thus Bob has an admissible move – he can color it with  $l$ . Suppose Bob colors  $e$  with  $j$ . Now the original game is on the matroids  $M_i|_{E \setminus \{e\}}$  for  $i \neq j$  and  $M_j/\{e\}$  on the set  $E \setminus \{e\}$ . For them the second condition of Theorem 2 holds since  $r_j(A)$  can possibly be lower only by one. Hence Alice has a winning strategy by inductive assumption.  $\square$

### 3. Modified indicated chromatic number

We consider a variant of the game from the previous section. Alice and Bob make alternative moves. If it is her turn, then as before, Alice indicates an uncolored yet element of  $E$ , and Bob colors it using a color from a fixed set  $C$ . While if it is Bob's turn their roles are swapped, now he indicates an uncolored yet element of  $E$ , and Alice colors it using a color from  $C$ . The other conditions of the game stay unchanged – elements of the same color must form an independent set and Alice wins if the whole matroid has been colored. The minimum size of the set of colors

$C$  for which Alice has a winning strategy we call a *modified indicated chromatic number* of  $M$ , and denote by  $\chi_i^{mod}(M)$ .

**THEOREM 4.** *Every loopless matroid  $M$  satisfies  $\chi_i^{mod}(M) = \chi(M)$ .*

As before in order to prove the above theorem we introduce a more general game. The theorem is a simple corollary of Theorem 5.

Let  $M_1, \dots, M_k$  be matroids on the same ground set  $E$ . We consider a game in which in each turn Bob decides which of the following two kinds of moves is played:

- (1) Alice indicates an uncolored yet element of  $E$  and Bob colors it using a color from  $\{1, \dots, k\}$ ,
- (2) Bob indicates an uncolored yet element of  $E$  and Alice colors it using a color from  $\{1, \dots, k\}$ .

Both players have to obey the rule that for each  $i$  elements colored with  $i$  form an independent set in the matroid  $M_i$ . If in some turn Alice or Bob does not have an admissible move the game ends. Alice wins if the whole matroid has been colored, while Bob wins if a partial coloring can not be properly extended.

**THEOREM 5.** *Let  $M_1, \dots, M_k$  be matroids on the same ground set  $E$ . Suppose there are sets  $V_1, \dots, V_k$ , such that  $V_i$  is independent in  $M_i$ , and  $V_1 \cup \dots \cup V_k = E$ . Then Alice has a winning strategy in the generalized modified indicated coloring game.*

**PROOF.** The proof goes by induction on the number of elements of  $E$ . For a set  $E$  consisting of only one element assertion clearly holds. Thus we assume that  $|E| > 1$  and that the assertion is true for all smaller sets.

If Bob decided that in the first turn second kind of move is played, then he points say  $e \in E$ . From the assumption  $e \in V_j$  for some  $j$ . Alice strategy is to color  $e$  with  $j$ . Now the remaining part is played on matroids  $M_i|_{E \setminus \{e\}}$  for  $i \neq j$  and  $M_j/\{e\}$ , so assumptions of the theorem are clearly satisfied and from inductive assumption Alice has a winning strategy.

If Bob decided that in the first turn first kind of move is played, then consider generalized indicated coloring game on matroids  $M_1, \dots, M_k$ . By Theorem 3 Alice has a winning strategy in this game. Suppose she indicates an element  $e \in V_i$ , and suppose Bob colors  $e$  with  $j$ . Now the generalized indicated coloring game is on the matroids  $M'_i = M_i|_{E \setminus \{e\}}$  for  $i \neq j$  and  $M'_j = M_j/\{e\}$ . Moreover, Alice still has a winning strategy, thus there exists a proper coloring in which elements of color  $i$  form an independent set in  $M'_i$ . For each  $i$  let  $U_i$  be the set of elements colored with  $i$ , we have  $U_1 \cup \dots \cup U_k = E \setminus \{e\}$ . Since the remaining part of generalized modified indicated coloring game is played also on matroids  $M'_i$ , this gives that the assumptions of the theorem are satisfied. Thus Alice has a winning strategy by inductive assumption.  $\square$

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES, ŚNIADECKICH 8, 00-956  
WARSZAWA, POLAND

THEORETICAL COMPUTER SCIENCE DEPARTMENT, FACULTY OF MATHEMATICS AND COMPUTER  
SCIENCE, JAGIELLONIAN UNIVERSITY, ŁOJASIEWICZA 6, 30-348 KRAKÓW, POLAND  
*E-mail address:* `michalason@gmail.com`